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Generalized Brownian Motion and Elasticity

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We introduce a family of stochastic processes which are a natural extension of Brownian motion to a tensor form. This allows us to solve a Dirichlet problem of linear elasticity obeying Lamé's equation, $[1-\nu(d-1)]\nabla^2\mathbf{V}(\mathbf{x})+\nabla[\nabla\cdot\mathbf{V}(x)]=0$.

KEY WORDS: Brownian motion; mathematical methods of elasticity.

1. INTRODUCTION

Since Brown in 1828[1] first described what has become known as random motion, this concept has performed a "random walk" through almost all fields of science. After an hesitating step in biology (Brown himself had the intuition of having discovered a "primitive molecule" of living matter!), the concept of Brownian motion proceeded to enter physics: diffusion (a few of Einstein's famous articles of 1905[2] are devoted to this problem), dispersion, heat transfer, electrostatics (Bachelier[3]), information theory and noise (Brillouin,[4] Shannon[5]), and, more recently, quantum mechanics (Nelson[6]) and field theory (e.g., G. Parisi). Another remarkable success of Brownian motion lies in pure and applied mathematics. The connection with harmonic analysis initiated by Bachelier[3] and developed by N. Wiener, A. Kolmogorov, and S. Kakutani[7] is still a field of active research. Amazingly enough, this somewhat universal concept seems to have avoided mechanics (aside from its recent implication in chaos and turbulence).

However, based on the strong similarities between the Laplace equation and the one governing the displacement field of an elastic solid, we propose here a random process built over a Brownian walk that allows one to solve this vector problem. It provides a natural generalization of

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Brownian motion to a tensor form and, to this extent, puts linear elasticity on a similar footing as electrostatics.

Section 2 recalls the basic analogy between Laplace's equation and Brownian motion. Section 3 defines the stochastic process. The vector nature of elasticity requires the definition of a tensor operator. In Section 3.1, the "*the first step*" introduces two basic tensor operators: identity and projection along a random vector. These two tensors will propagate vector-valued information in the same way as a usual random walk transports scalar information. Section 3.2, "*the walk*," considers the iteration of these basic operators and gives the limit processes in Fourier space, thanks to the central limit theorem. Section 3.3, "*the continuous evolution*," investigates the meaning of these continuous operators through evolution equations expressed in terms of infinitesimal generators. Now we have two differential equations at our disposal. In order to recover Lamé's equation in all its generality (i.e., for any Poisson ratio) we need some "*freedom*" (Section 3.4). We will show that a mere linear combination of the two fundamental processes gives an evolution equation related to Lamé's equation. Up to this point, the problem is treated in an infinite space; Section 3.5, "*boundaries*," restricts our random walk with a boundary rule that enables us to remain in a compact domain and takes into account the boundary condition of a Dirichlet problem (the displacement field at the boundary is specified). Finally, Section 3.6, "*simplicity*," gets rid of the evolution operator, unnecessary for our initial purpose, and answers the basic point of interest. The solution of any Dirichlet problem of linear elasticity on a compact domain can be obtained through a stochastic process. The last two points should be considered as suggestive rather than an exact mathematical proof.

The reader interested in the result only may skip Section 3, since the results are reported in Section 4. On the other hand, the reader familiar with the classic analogy between potential theory and Brownian motion may find it helpful, all though Section 3, to consider the scalar reduction of any expression involving operator I.

2. PRELIMINARY CONSIDERATIONS

In the following the sign $\otimes$ will denote the tensor product and $\cdot$ the contraction of two tensors. $|\mathbf{k}|^2$ is the square norm of vector $\mathbf{k}$.

Let us recall the basic results arising from the analogy between Brownian motion and potential theory: Consider the following problem:

$$\nabla^2 V(\mathbf{x}) = 0 \qquad \text{for all} \quad \mathbf{x} \in D^0 \tag{1}$$

$$V(\mathbf{x}) = I(\mathbf{x}) \qquad \text{for all} \quad \mathbf{x} \in \partial D \tag{2}$$

where D is a compact domain of R^d, ∂D its boundary, and D^0 its open region. If M is a point of D^0, W_M a random walk starting from M, and $P(W_M)$ the first intersection point between W_M and ∂D, then the potential $V(M)$ in M is given by

$$V(M) = \langle U(P(W_M)) \rangle_W \tag{3}$$

the average $\langle \cdots \rangle_W$ being taken over all random walks W_M. Equivalently, Eq. (3) can be written as:

$$V(M) = \int_{\partial D} U(\mathbf{x})\, d\mu(\mathbf{x}) \tag{3'}$$

where $d\mu(\mathbf{x})$ is a measure on ∂D, defined as an exit probability at point $\mathbf{x}$ for a random walk starting at M (Fig. 1).

The displacement field $\mathbf{V}(\mathbf{x})$ of an elastic solid whose Poisson ratio is v is a solution of Lamé's equation (4) in a d-dimensional space. The equivalent problem to Eqs. (1) and (2) is

$$[1 - v(d-1)]\, \nabla^2 \mathbf{V}(\mathbf{x}) + \nabla[\nabla \cdot \mathbf{V}(\mathbf{x})] = 0 \qquad \text{for all} \quad \mathbf{x} \in D^0 \tag{4}$$

$$\mathbf{V}(\mathbf{x}) = \mathbf{U}(\mathbf{x}) \qquad \text{for all} \quad \mathbf{x} \in \partial D \tag{5}$$

with index notation and Einstein implicit summation, (4) reads

$$[1 - v(d-1)]\; V_{i,jj}(\mathbf{x}) + V_{j,ji}(\mathbf{x}) = 0 \tag{4'}$$

Lamé's equation shares a lot of properties with Laplace's (1): Both are linear, elliptic, second-order differential equations. However, the vector

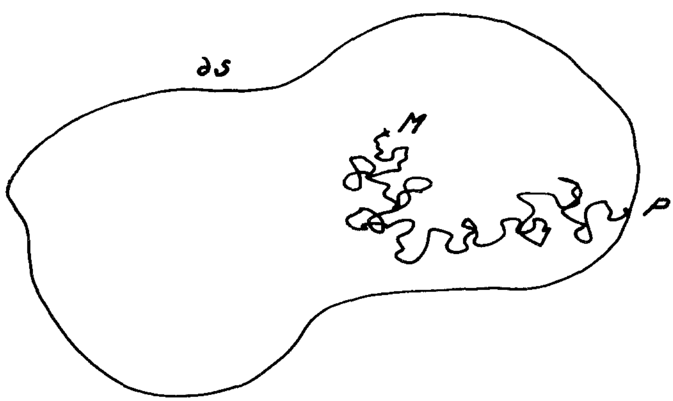

Fig. 1. The potential $V(M)$ in M is the average of $V(P)$ over P, where P is the first intercept of a random walk started in M with the boundary ∂S where the potential is fixed.

nature of the former prevents us from applying directly classical results known for the scalar case (Laplace's and related problems). In this paper we propose a stochastic process—a generalization of Brownian motion—which can solve the problem defined by Eqs. (4) and (5) in a way similar to Eq. (3) or Eq. (3′). The relation corresponding to Eq. (3′) is then

$$\mathbf{V}(M) = \int_{\partial D} \Pi_M(\mathbf{x}) \cdot \mathbf{U}(\mathbf{x}) \, d\mu'(\mathbf{x}) \tag{5'}$$

$\Pi_M(\mathbf{x})$ is a second-order tensor, and $d\mu'(\mathbf{x})$ a measure on ∂D.

3. CONSTRUCTION

One can consider the Brownian motion as the limit of a series of random steps of length r uniformly distributed in space. The limit to consider is the following: the number of steps n tends to infinity, the length r tends to 0, while the combination nr^2 has a finite limit called t (usually a time parameter). In a parallel way, we will define now a stochastic process that constructs a tensor during its progression.

3.1. The First Step

Let $\mathbf{e}$ be a random unit vector of R^d, whose probability density is uniform on the surface of a unit sphere. We consider two second-order tensors: the identity I and the projection operator along $\mathbf{e}$: $\mathbf{e} \otimes \mathbf{e}$. Using index notations, we can write

$$\mathsf{I}_{ij} = \delta_{ij} \qquad \text{(Kronecker symbol)} \tag{6}$$

$$(\mathbf{e} \otimes \mathbf{e})_{ij} = e_i e_j \tag{7}$$

Let us call $\mathsf{P}(\mathbf{e}) = d(\mathbf{e} \otimes \mathbf{e})$, where d is the space dimensionality and is introduced here for normalization purpose. These two tensors are naturally associated with two operators I_r and P_r, which transform any vector field $\mathbf{A}(\mathbf{x})$ in the following way:

$$(\mathsf{I}_r \mathbf{A})(\mathbf{x}) = \langle \mathsf{I}(\mathbf{e}) \cdot \mathbf{A}(\mathbf{x} + r\mathbf{e}) \rangle_e = \langle \mathbf{A}(\mathbf{x} + r\mathbf{e}) \rangle_e \tag{8}$$

$$(\mathsf{P}_r \mathbf{A})(\mathbf{x}) = \langle \mathsf{P}(\mathbf{e}) \cdot \mathbf{A}(\mathbf{x} + r\mathbf{e}) \rangle_e = d \langle [\mathbf{e} \cdot \mathbf{A}(\mathbf{x} + r\mathbf{e})] \cdot \mathbf{e} \rangle_e \tag{9}$$

$$= d \langle \mathbf{A}_j(\mathbf{x} + r\mathbf{e}) \cdot \mathbf{e}_j \cdot \mathbf{e}_i \rangle_e \tag{9'}$$

The symbol $\langle \cdots \rangle_e$ means that we have considered averages over the random vector $\mathbf{e}$. Thus, $(\mathsf{I}_r \mathbf{A})(\mathbf{x})$ represents the average of the vectors $\mathbf{A}(\mathbf{y})$ for all $\mathbf{y}$ sitting on a sphere of radius r and center $\mathbf{x}$. The term $(\mathsf{P}_r \mathbf{A})(\mathbf{x})$ is equal to the average of the projection of $\mathbf{A}(\mathbf{y})$ onto the direction $(\mathbf{y} - \mathbf{x})$ for all $\mathbf{y}$ on the same sphere (see Fig. 2).

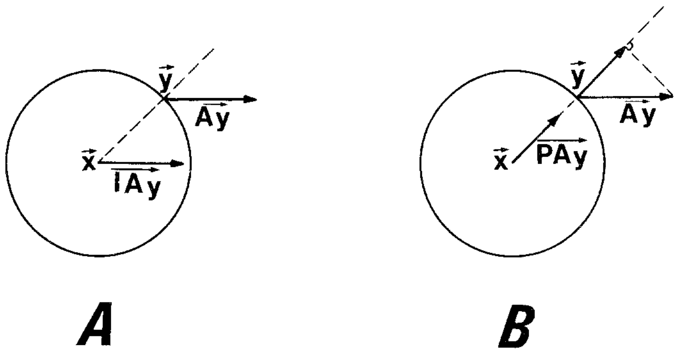

Fig. 2. The operators I and P applied to a vector field $\mathbf{A}(\mathbf{x})$ give the following result: (A) $(\mathsf{I}_r\mathbf{A})(\mathbf{x})$ is the average of $\mathbf{A}(\mathbf{y})$ where $\mathbf{y}$ lies on a sphere of radius r centered in $\mathbf{x}$. (B) $(\mathsf{P}_r\mathbf{A})(\mathbf{x})$ is the average of the projection of $\mathbf{A}(\mathbf{y})$ on the direction $(\mathbf{y}-\mathbf{x})$ (up to a numerical factor d).

We have a normalization property for I_r and P_r

$$\langle \mathsf{I} \rangle_e = \mathsf{I} \tag{10}$$

$$\langle \mathsf{P}(\mathbf{e}) \rangle_e = \mathsf{I} \tag{11}$$

(The numerical factor d in the definition of P had been introduced for that purpose.) If $\mathbf{A}(\mathbf{x})$ is a constant field $\mathbf{A}$, then

$$(\mathsf{I}_r\mathbf{A}) = \mathbf{A} \tag{12}$$

$$(\mathsf{P}_r\mathbf{A}) = \mathbf{A} \tag{13}$$

In fact, we have a stronger property: For every vector field $\mathbf{T}(\mathbf{x})$ that is a sum of a constant vector and a rotation tensor applied on $\mathbf{x}$, we have

$$(\mathsf{I}_r\mathbf{T})(\mathbf{x}) = \mathbf{T}(\mathbf{x}) \tag{14}$$

$$(\mathsf{P}_r\mathbf{T})(\mathbf{x}) = \mathbf{T}(\mathbf{x}) \tag{15}$$

The proof of such an invariance is straightforward if, for any given point $\mathbf{x}$, we decompose $\mathbf{T}(\mathbf{y})$ into a translation part $\mathrm{T}(\mathrm{x})$ and a rotation $\mathbf{R}_x(\mathbf{y})$ around $\mathbf{x}$; then

$$\mathbf{T}(\mathbf{y}) = \mathbf{T}(\mathbf{x}) + \mathbf{R}_x(\mathbf{y}) \tag{16}$$

and symmetry gives

$$(\mathsf{I}_r\mathbf{R}_x)(\mathbf{x}) = 0 \tag{17}$$

$$(\mathsf{P}_r\mathbf{R}_x)(\mathbf{x}) = 0 \tag{18}$$

The properties (12), (13) and (17), (18), together with the decomposition, prove the results. We will use this invariance for our purpose: Such a vector field $\mathbf{T}(\mathbf{x})$ represents the strain-free displacement field of a rigid. Any such field will satisfy Eq. (4) [in the same way, any constant potential satisfies Eq. (1)]. Some elementary algebra gives the first "moments" of the operators I_r and P_r, through the average of the following tensors:

$$[\langle \mathsf{I}(\mathbf{e}) \otimes \mathbf{e} \rangle_e]_{ijk} = \langle \delta_{ij} e_k \rangle_e = 0 \tag{19}$$

$$[\langle \mathsf{P}(\mathbf{e}) \otimes \mathbf{e} \rangle_e]_{ijk} = d\langle e_i e_j e_k \rangle_e = 0 \tag{20}$$

$$[\langle \mathsf{I}(\mathbf{e}) \otimes \mathbf{e} \otimes \mathbf{e} \rangle_e]_{ijkl} = \langle \delta_{ij} e_k e_l \rangle_e = (\delta_{ij} \delta_{kl})/d \tag{21}$$

$$[\langle \mathsf{P}(\mathbf{e}) \otimes \mathbf{e} \otimes \mathbf{e} \rangle_e]_{ijkl} = d\langle e_i e_j e_k e_l \rangle_e = (\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})/(d+2) \tag{22}$$

3.2. The Walk

As in the case of a Brownian walk, we will iterate elementary steps and consider the operators I_r^n and P_r^n. Let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ be n independent vectors,

$$\mathsf{I}^n(\mathbf{e}_1, \ldots, \mathbf{e}_n) = \mathsf{I} \tag{23}$$

$$\begin{aligned} \mathsf{P}^n(\mathbf{e}_1, \ldots, \mathbf{e}_n) &= \mathsf{P}(\mathbf{e}_1) \cdot \mathsf{P}(\mathbf{e}_2) \cdot \cdots \cdot \mathsf{P}(\mathbf{e}_n) \\ &= d^n (\mathbf{e}_1 \cdot \mathbf{e}_2)(\mathbf{e}_2 \cdot \mathbf{e}_3) \cdots (\mathbf{e}_{n-1} \cdot \mathbf{e}_n)(\mathbf{e}_1 \otimes \mathbf{e}_n) \end{aligned} \tag{24}$$

The tensor part of the expression (24) takes into account only the first and last steps. The scalar factor includes the whole history of the walk through the scalar product of all successive pairs of neighboring steps. However, our process is still Markovian: the conditional law of P^{n+1} knowing the whole history $\mathsf{P}^1, \mathsf{P}^2, \ldots, \mathsf{P}^n$ is the same as knowing only the last step P^n. The operators I_r^n and P_r^n act on a vector field $\mathbf{A}(\mathbf{x})$ according to

$$(\mathsf{I}_r^n \mathbf{A})(\mathbf{x}) = \langle \mathbf{A}(\mathbf{x} + r\mathbf{e}_1 + r\mathbf{e}^2 + \cdots + r\mathbf{e}_n) \rangle_{(e)} \tag{25}$$

$$(\mathsf{P}_r^n \mathbf{A})(\mathbf{x}) = \langle \mathsf{P}^n(\mathbf{e}_1, \ldots, \mathbf{e}_n) \cdot \mathbf{A}(\mathbf{x} + r\mathbf{e}_1 + r\mathbf{e}_2 + \cdots + r\mathbf{e}_n) \rangle_{(e)} \tag{26}$$

In order to analyze the properties of P_r^n, it is useful to evaluate the spatial Fourier transform of I_r, P_r, and $\mathbf{A}(\mathbf{x})$. Let us define

$$\underline{\mathsf{P}}_r(\mathbf{k}) < \langle \mathsf{P}(\mathbf{e}) \exp(-i\mathbf{k} \cdot r\mathbf{e}) \rangle_e \tag{27}$$

If $\underline{A}(\mathbf{k})$ is the Fourier transform of $\mathbf{A}(\mathbf{x})$

$$\underline{A}(\mathbf{k}) = \int \exp(-i\mathbf{k} \cdot \mathbf{x})\, \mathbf{A}(\mathbf{x})\, d\mathbf{x}$$

then Eq. (9) reads

$$(\underline{\mathsf{P}}_r\underline{A})(\mathbf{k}) = \underline{\mathsf{P}}_r(\mathbf{k}) \cdot \underline{A}(\mathbf{k}) \tag{28}$$

Our calculation (19)–(22) allows us to expand the tensors $\mathsf{I}_r(\mathbf{k})$ and $\underline{\mathsf{P}}_r(\mathbf{k})$:

$$\begin{aligned}\underline{I}_r(\mathbf{k}) &= \mathsf{I} - r^2 \langle \mathsf{I} \otimes \mathbf{e} \otimes \mathbf{e} \rangle_e \cdot\cdot (\mathbf{k} \otimes \mathbf{k})/2 + O(k^4 r^4) \\ &= \mathsf{I} - r^2 |\mathbf{k}|^2 \mathsf{I}/2d + O(k^4 r^4) \\ \underline{\mathsf{P}}_r(\mathbf{k}) &= \mathsf{I} - r^2 \langle \mathsf{P}(\mathbf{e}) \otimes \mathbf{e} \otimes \mathbf{e} \rangle_e \cdot\cdot (\mathbf{k} \otimes \mathbf{k})/2 + O(k^4 r^4)\end{aligned} \tag{29}$$

$$= \mathsf{I} - r^2(|\mathbf{k}|^2 \mathsf{I} + 2\mathbf{k} \otimes \mathbf{k})/2(d+2) + O(k^4 r^4) \tag{30}$$

We need the nth power of $\underline{\mathsf{I}}_r(\mathbf{k})$ and $\underline{\mathsf{P}}_r(\mathbf{k})$ in the limit of n going to infinity, r to 0, and the product nr^2 tending to t. Paralleling the demonstration of the central-limit theorem gives us the following limiting forms of $\underline{\mathsf{I}}_r^n(\mathbf{k})$ and $\underline{\mathsf{P}}_r^n(\mathbf{k})$:

$$\begin{aligned}\lim \underline{\mathsf{I}}_r^n(\mathbf{k}) &= \underline{\mathsf{I}}^t(\mathbf{k}) \\ \lim \underline{\mathsf{P}}_r^n(\mathbf{k}) &= \underline{\mathsf{P}}^t(\mathbf{k})\end{aligned} \tag{31}$$

where

$$\underline{\mathsf{I}}^t(\mathbf{k}) = \exp(-t|\mathbf{k}|^2 \mathsf{I}/2d) = \mathsf{I} \exp(-t|\mathbf{k}|^2/2d) \tag{32}$$

$$\underline{\mathsf{P}}^t(\mathbf{k}) = \exp[-t(|\mathbf{k}|^2 \mathsf{I} + 2\mathbf{k} \otimes \mathbf{k})/2(d+2)] \tag{33}$$

$\underline{\mathsf{P}}^t(\mathbf{k})$ is the Fourier transform of the equivalent to the transition probabilities of a usual Brownian walk.

The tensor nature of I^t is no more than an artefact: The process I can be decoupled into d independent Brownian walks. We have obtained one of our final tools. The operator $\mathsf{P}^t(\mathbf{x})$ [given in (32) through its Fourier transform $\underline{\mathsf{P}}^t(\mathbf{k})$)] is a key for our purpose: we will investigate below the meaning of

$$(\mathsf{P}^t\mathbf{A})(\mathbf{x}) = \int \mathsf{P}^t(\mathbf{x} - \mathbf{y}) \cdot \mathbf{A}(\mathbf{y})\, d\mathbf{y} \tag{34}$$

through its time derivative and finally we will find a way to solve an evolution equation related to Lamé's equation in an infinite domain.

3.3. Continuous Evolution

The time derivative $\underline{\imath}(\mathbf{k})$ of the transformation $\mathbf{A}(\mathbf{x}) \to (\mathsf{I}^t\mathbf{A})(\mathbf{x})$ in Fourier space, after having taken the expectation, is given by

$$\underline{\imath}(\mathbf{k}) = \lim_{t \to 0} [\underline{\mathsf{I}}^t(\mathbf{k}) - \mathbb{1}]/t = |\mathbf{k}|^2 \mathbb{1}/2d \tag{35}$$

or

$$\underline{\iota}(\mathbf{x}) = (\nabla^2)/2d \tag{36}$$

Respectively, for $\mathbf{A}(\mathbf{x}) \rightarrow (\mathsf{P}^t\mathbf{A})(\mathbf{x})$, the time derivative $\underline{\mathbf{\Pi}}(\mathbf{k})$, still after expectation, reads

$$\begin{aligned}\underline{\mathbf{\Pi}}(\mathbf{k}) &= \lim_{t\to 0}[\underline{\mathsf{P}}^t(\mathbf{k}) - \underline{\mathsf{P}}^0(\mathbf{k})]/t \\ &= \lim_{t\to 0}[\underline{P}^t(\mathbf{k}) - \mathbb{1}]/t \\ &= (|\mathbf{k}|^2 \mathsf{I} + 2\mathbf{k}\otimes\mathbf{k})/2(d+2)\end{aligned} \tag{37}$$

or

$$\begin{aligned}\mathbf{\Pi}(\mathbf{x}) &= (\nabla^2 + 2\nabla\otimes\nabla)/2(d+2) \\ &= \{\nabla^2 + 2\,\mathrm{grad}[\mathrm{div}(\cdot)]\}/2(d+2)\end{aligned} \tag{38}$$

Therefore, we have the following result: $(\mathsf{I}^t\mathbf{A})(\mathbf{x})$ is a solution of

$$\partial\mathbf{V}(\mathbf{x}, t)/\partial t = \nabla^2\mathbf{V}/2d \tag{39}$$

$$\mathbf{V}(\mathbf{x}, 0) = \mathbf{A}(\mathbf{x}) \tag{40}$$

whereas $(\mathsf{P}^t\mathbf{A})(\mathbf{x})$ satisfies

$$\partial\mathbf{V}(\mathbf{x}, t)/\partial t = [\nabla^2\mathbf{V} + 2\nabla\cdot\mathbf{V})]/2(d+2) \tag{41}$$

$$\mathbf{V}(\mathbf{x}, 0) = \mathbf{A}(\mathbf{x}) \tag{42}$$

Lamé's squation is more general than the rhs of (39) or (41), but we will recover it through a linear combination of our two processes.

3.4. Freedom

The stationary solution of Eqs. (41) and (42) is a solution of Lamé's equation (4) if $\nu = 1/2(d-1)$. In order to recover Lamé's equation for any physical situation $[-1 \leqslant \nu \leqslant 1/(d-1)]$, we must consider another stochastic process: In fact, a mere linear combination of I^t and P^t provides us with our goal. At each step, let us choose the tensor $\mathsf{Q} = \alpha\mathsf{I} + (1-\alpha)\mathsf{P}$. The α and $1-\alpha$ could also be considered as probabilities of choosing I or P. However, this additional probabilistic element does not provide any novel or interesting feature. Therefore, from a practical point of view, we prefer to consider the average $\alpha\mathsf{I} + (1-\alpha)\mathsf{P}$. Then,

$$\langle \mathsf{Q}(\mathbf{e}) \rangle = \mathsf{I} \tag{43}$$

$$\langle \mathsf{Q}(\mathbf{e}) \otimes \mathbf{e} \rangle = 0 \tag{44}$$

$$\begin{aligned}\langle \mathsf{Q}(\mathbf{e}) \otimes \mathbf{e} \otimes \mathbf{e} \rangle &= [\alpha/d + (1-\alpha)/d(d+2)]\, \delta_{ij}\delta_{kl} \\ &\quad + (1-\alpha)/d(d+2)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\end{aligned} \tag{45}$$

Following our derivation, we obtain

$$Q'(\mathbf{k}) = \exp\{-t/2d(d+2)[(\alpha d+\alpha+1)|\mathbf{k}|^2 + 2(1-\alpha)\mathbf{k}\otimes\mathbf{k}]\} \quad (46)$$

The corresponding evolution problem for $(Q'\mathbf{A})(\mathbf{x})$ is

$$\partial\mathbf{V}(\mathbf{x}, t)/\partial t = [(\alpha d+\alpha+1)\nabla^2\mathbf{V} + 2(1-\alpha)\nabla(\nabla\mathbf{V})]/2d(d+2) \quad (47)$$

$$\mathbf{V}(\mathbf{x}, 0) = \mathbf{A}(\mathbf{x}) \quad (48)$$

Identification of (47) and (4) leads to

$$1-(d-1)\nu = (\alpha d+\alpha+1)/2(1-\alpha)$$

or

$$\alpha = [1-2(d-1)\nu]/[d+3-2(d-1)\nu] \quad (49)$$

valid for $-1 \leqslant \nu \leqslant 1/2(d-1)$. The remaining cases [i.e., $1/2(d-1) \leqslant \nu \leqslant 1/(d-1)$] will be reached by the choice $\alpha\mathsf{I} - (1-\alpha)\mathsf{P}$. A similar calculation gives

$$\alpha = [2\nu(d-1)-1]/[2\nu(d-1)+d+1] \quad (50)$$

Now we have an evolution process that leads to the solution of Lamé's equation in the limit of infinite time. We will now suggest how one can incorporate the boundaries of a finite domain in our present formulation.

3.5. And Boundaries

In this section and the following we present (rather than prove) two additional tools that enables us to deal with realistic problems. The way we described our processes, I, P, or Q is well defined in the case of an infinite domain. If we deal with a finite domain D, we need new rules. Suppose we stop whenever our walk encounters the boundary ∂D; then, for any point $\mathbf{x}$ lying within D^0, we may consider a sphere B centered around $\mathbf{x}$ and of radius ε that is within D^0. Inside B, the process is not altered by our boundary rule, so that Eqs. (39), (41), and (47) still hold. But for points $\mathbf{y}$ on ∂D, $\mathbf{V}(\mathbf{y}, t) = \mathbf{V}(\mathbf{y}, 0) = \mathbf{A}(\mathbf{y})$. Finally, we suggest that the stochastic processes considered at the beginning of this paper, complemented by this straightforward rule, will solve the following problem:

$$\partial\mathbf{V}(\mathbf{x}, t)/\partial t = [1-\nu(d-1)]\nabla^2\mathbf{V}(\mathbf{x}) + \nabla[\nabla\cdot\mathbf{V}(\mathbf{x})] \quad \text{for all } \mathbf{x}\in D^0 \quad (51)$$

$$\mathbf{V}(\mathbf{x}, t) = \mathbf{A}(\mathbf{x}) \quad \text{for all } \mathbf{x}\in\partial D \quad (52)$$

$$\mathbf{V}(\mathbf{x}, 0) = \mathbf{A}(\mathbf{x}) \quad \text{for all } \mathbf{x}\in D \quad (53)$$

Now we know how to solve Eqs. (4) and (5):

1. We construct an initial field $\mathbf{A}(\mathbf{x})$ in D so that $\mathbf{A}(\mathbf{x})|_{\partial D} = \mathbf{U}(x)$, $\mathbf{A}(x)$ being arbitrary inside D^0.
2. We consider the operator Q^t (the Poisson ratio ν is given).
3. We apply it on $\mathbf{A}(\mathbf{x})$ and consider the limit $(\mathsf{Q}^{\infty}\mathbf{A})(\mathbf{x})$.

This field is a solution of Eqs. (4) and (5).

3.6. Simplicity

However, the first point is rather awkward. It introduces an arbitrary initial field [$\mathbf{A}(\mathbf{x})$ in D^0], which should not affect the final result, since we consider the stationary solution of Eqs. (51) and (53). However, instead of considering the whole evolution problem, which gives us too much information, we may use the fact that in the limit of $t \to \infty$, almost all walks will have encountered the boundary. The number of remaining "free" walks that did not stick to ∂D at time t (and so never encountered ∂D up to time t) decays exponentially with time in a compact domain D. So let us forget about the value assigned to the initial field $\mathbf{A}(\mathbf{x})$ inside D^0. If, for each walk, we stop as soon as we meet ∂D, then we can compute the solution of Eqs. (4) and (5) in the whole domain D. We propose that the field obtained in this way is the stationary solution of Eq. (51) (see the Appendix for details). This concludes our initial question:

To any random walk W_M starting in M and reaching ∂D for the first time in $P(W_M)$, we can associate a tensor $\mathsf{T}(W_M)$ [defined below in Eqs. (55) and (56)]. The displacement field $\mathbf{V}(M)$ in M will be given by

$$\mathbf{V}(M) = \langle \mathsf{T}(W_M) \cdot \mathbf{U}(P(W_M)) \rangle_{(w)} \tag{54}$$

the average $\langle \cdot \rangle_{(w)}$ being taken over all random walks. This last expression is to be compared with Eqs. (3) and (5).

4. RESULTS

Let us summarize the detailed discretized procedure we can use to solve Eqs. (4) and (5):

1. For any given point M, let us consider an elementary step of length r in an arbitrary direction $\mathbf{e}_1$. A tensor $\mathsf{T}(\mathbf{e}_1)$ is associated to this step:

$$\mathsf{T}(\mathbf{e}_1) = \{[1 - 2(d-1)\nu]\mathsf{I} + (d+2)/d\mathbf{e}_1 \otimes \mathbf{e}_1\}/[d + 3 - 2(d-1)\nu] \tag{55}$$

if $-1 \leqslant \nu \leqslant 1/2(d-1)$,

$$\mathsf{T}(\mathbf{e}_1) = \{[2(d-1)\nu - 1]\mathsf{I} - (d+2)/d\mathbf{e}_1 \otimes \mathbf{e}_1\}/[d+1+2(d-1)\nu] \tag{56}$$

if $1/2(d-1) \leqslant \nu \leqslant 1/(d-1)$. Let us call $\mathsf{T} = \mathsf{T}(\mathbf{e}_1)$.

2. After another random step $r\mathbf{e}_i$ has been chosen, the tensor T is replaced by $\mathsf{T} \cdot \mathsf{T}(\mathbf{e}_i)$. This second point is repeated until the end of the walk $M + \sum r\mathbf{e}_i$ crosses the boundary ∂D.

3. If P denotes the intersection point between the walk and ∂D, then we compute $\mathsf{T} \cdot \mathbf{U}(P)$.

4. Now we average $\mathsf{T} \cdot \mathbf{U}(P)$ over different random walks.

The resulting vector converges toward the solution of Eqs. (4) and (5) as the length of each step r tends to 0.

5. CONCLUSION

We have introduced a tensor stochastic process related to linear elasticity in order to establish a connection between Brownian motion and potential theory. This adds a novel relation between elasticity and electricity. Other similarities have already proven to be very useful tools (for the application of potential concepts to elasticity and the use of Green's functions see, e.g., Boussinesq[8]; for a review of analytical results see Solomon[9]).

One very current opening can be found in the field of fracture in elastic solids. A very attractive model, initially introduced to describe the growth of colloids, has been widely studied recently: diffusion-limited aggregation.[10,11] A cluster is grown from particles performing random walks until they touch and stick to the cluster. The patterns obtained show a self-similar structure, which can be characterized by a fractal dimension strictly smaller than the Euclidian dimension of the embedding space. Such objects are also obtained in other processes, such as invasion front in porous media, in dielectric breakdown (Pietronero) (see Ref. 12 for a review), and in the development of Saffman–Taylor instabilities in hydrodynamics. It is possible to show that an elastic fracture problem is related to the dual problem of diffusion-limited aggregation (related means that Lamé's equation has to be considered instead of Laplacé's). Thus, our stochastic process might be a natural way to handle the problem of the structure of a fracture pattern.

Moreover, although we restrict ourselves to Lamé's equation in the framework of linear elasticity, it is worth noting that such an approach can be applied to other kinds of linear elliptic differential equations. It may be interesting to develop this aspect, as well as other mathematical points (convergence of the stochastic process, well-defined mathematical framework, etc.).

APPENDIX

Let us consider the resulting field $\mathbf{U}(\mathbf{x})$ of the process defined in Section 6 with boundary condition $\mathbf{U}(\mathbf{x}) = \mathbf{A}(\mathbf{x})$. Let this field be the initial field of the evolution problem defined in the same situation:

$$\partial \mathbf{V}(\mathbf{x}, t)/\partial t = [1 - \nu(d-1)]\, \nabla^2 \mathbf{V}(\mathbf{x}) + \nabla[\nabla \cdot \mathbf{V}(\mathbf{x})] \quad \text{for all } x \in D^0 \tag{A1}$$

$$\mathbf{V}(\mathbf{x}, t) = \mathbf{A}(\mathbf{x}) \quad \text{for all } x \in \partial D \tag{A2}$$

$$\mathbf{V}(\mathbf{x}, 0) = \mathbf{U}(\mathbf{x}) \quad \text{for all } x \in D \tag{A3}$$

$\mathbf{V}(\mathbf{x}, t)$ will be the average over all the walks $W(\mathbf{x}, \mathbf{y})$ starting in $\mathbf{x}$ at time $t = 0$ ($\mathbf{y}$ denotes the end of the walk at time t) of the tensor $\mathrm{Q}'(W(\mathbf{x}, \mathbf{y}))$ applied to $\mathbf{V}(\mathbf{y}, 0) = \mathbf{U}(\mathbf{y})$. But $\mathbf{U}(\mathbf{y})$ is the average over all the walks $W(\mathbf{y}, \mathbf{z})$ (starting in $\mathbf{y}$ and ending on the boundary $\mathbf{z}$) of the tensor $\mathrm{Q}(W(\mathbf{y}, \mathbf{z}))$ applied to $\mathbf{A}(\mathbf{z})$. So

$$\mathbf{V}(\mathbf{x}, t) = \langle \mathrm{Q}'(W(\mathbf{x}, \mathbf{y})) \cdot \langle \mathrm{Q}(W(\mathbf{y}, \mathbf{z})) \rangle \cdot \mathbf{A}(\mathbf{z}) \rangle \tag{A4}$$

$$= \langle \mathrm{Q}(W(\mathbf{x}, \mathbf{z})) \cdot \mathbf{A}(\mathbf{z}) \rangle \tag{A5}$$

thanks to the facts that the two pieces of walk $W(\mathbf{x}, \mathbf{y})$ and $W(\mathbf{y}, \mathbf{z})$ connected one to the other reproduce a random walk from $\mathbf{x}$ to $\mathbf{z}$ with the same statistics when integrated over all $\mathbf{y}$, and that the contraction of the two tensors Q gives the equivalent Q for the connected walk. So we recover in (A5) the property that $\mathbf{V}(\mathbf{x}, t)$, being the stationary solution of Eq. (51–53), is the solution of Eq. (4, 5). This appendix should be considered as suggestive rather than a rigorous proof.

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